BRIEF REPORTS

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than four printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.

Reversible fluctuation rectifier

I. M. Sokolov
*Laboratoire des Milieux Désordonnées et Hétérogènes, Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris, France and Theoretische Polymerphysik, Universität Freiburg, Hermann-Herder Strasse 3, D-79104 Freiburg i.Br., Germany**
(Received 27 April 1999)

The analysis of a Feynman's ratchet system [J. M. R. Parrondo and P. Español, Am. J. Phys. **64**, 1125 (1996)] and of its electrical counterpart, a diode engine [I. M. Sokolov, Europhys. Lett. **44**, 278 (1998)] has shown that "fluctuation rectifiers" consisting of a nonlinear element (ratchet, diode) and a linear element (vane, resistor) kept at different temperatures always show efficiency smaller than the Carnot value, thus indicating the irreversible mode of operation. We show that this irreversibility is not intrinsic for a system in simultaneous contact with two heat baths at different temperatures and that a fluctuation rectifier can work reversibly. This is illustrated by a model with two diodes switched in opposite directions, where the Carnot efficiency is achieved when backward resistivity of the diodes tends to infinity. [S1063-651X(99)15510-7]

PACS number(s): 05.70.Ln, 05.40.−a

Recent interest in the way of operation of molecular motors, cell pumps, etc. (see Refs. [1–5] for a review) has also revived interest in general aspects of the thermodynamics of irreversible processes, especially in the problem of energetics and the efficiency of nonequilibrium engines [6–8]. Interestingly enough, many of the real biological systems have been found to be closely related to the ratchet-and-pawl engine, considered in Chap. 46 of [9]. This appliance was invented *ad hoc* to provide students with a qualitative explanation of Carnot's formula and the second law of thermodynamics. The engine is a fluctuation rectifier, consisting of an axle with vanes on one of its ends and a ratchet on the other one. Contrary to typical, cyclic heat engines (like a steam engine or an Otto motor) and to the ratchet engines of Refs. [10, 11], which instantaneously are in contact with either a hot or cold heat bath, Feynman's engine works continuously, being simultaneously in contact with both reservoirs. The analysis of Parrondo and Español [12] shows that the Feynmann ratchet-and-pawl engine is essentially irreversible, since the mechanical link between the vanes and the ratchet implies that the reservoirs are not thermally isolated. This mechanical coupling induces, via fluctuations, a heat transfer-between the reservoirs, even under the no-work condition. Thus, the ratchet engine differs considerably from reversible, cyclically working machines. Such machines give rise to no heat exchange at zero power and, moreover, achieve at zero power their highest thermal efficiency, namely, the Carnot one, $\eta_{\mathrm{Carnot}}=(T_1-T_2)/T_1$.

In Ref. [13] explicit calculations for a diode engine are performed. The diode engine of Ref. [13] can be considered as a strongly simplified version of Feynman's ratchet, where all internal degrees of freedom are described in a mean-field manner by a nonlinear response function. The diode engine is even an older species than Feynmann's appliance: it is related to the Brillouin diode circuit [14] discussed in 1950 in order to explain the impossibility of the rectification of thermal fluctuations by an isothermal nonlinear system. The calculations show that a diode engine behaves just like its mechanical prototype: it transports heat also under a no-work condition. Its efficiency is zero at zero power and achieves its maximal value at finite current. The value of this maximal efficiency is, of course, lower than the Carnot one.

In the present work we address the question of whether an engine that is simultaneously in contact with two reservoirs is inevitably irreversible. As we proceed to show, this is not the case.

All "realistic" heat engines, including those working cyclically and applying reversible processes (Carnot, Stirling, etc.), are irreversible due to inevitable losses connected with the nonideality of the elements used. We are hence to define the rules that help to distinguish between this "technical" irreversibility and the intrinsic one inherent in the mode of operation. These rules are as follows: after the construction of the engine is fixed (which implies fixing the functional form of the equations of state, types of response, etc.), one can arbitrarily change the values of specific parameters and kinetic coefficients (unless they do not contradict each other and are not forbidden by thermodynamic inequalities). This includes the possibility of considering the limiting values of zero or infinity. If for some values of parameters the engine achieves the Carnot efficiency, it is considered as principally reversible, independent of the practical accessibility of these values. These rules fully conform with our general belief that

*Permanent address.

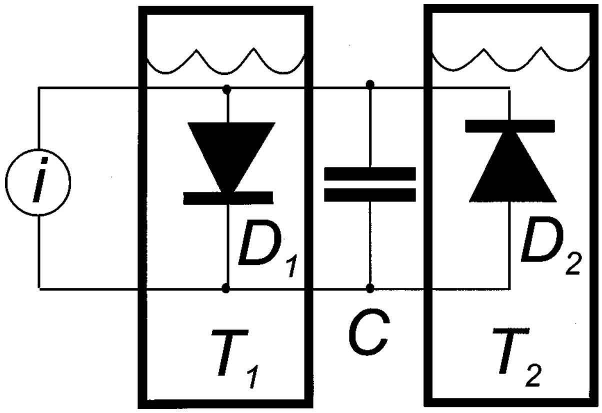

FIG. 1. Model system under consideration: the heat engine consisting of two diodes, D_1 (kept at temperature T_1) and D_2 (kept at temperature T_2), and a capacitor C, works against an outer current generator.

Carnot's engine is reversible, although we do not have at our disposal any ideal gas, or frictionless piston, or cylinder with zero heat capacity.

Our system, see Fig. 1, is a variant of a system considered in Ref. [13], where, instead of a resistor R, the second diode D_2 is included. To simplify calculations, the diodes are considered to be identical. The system produces work by working against a current generator (an abstraction often used in the theory of electric circuits and corresponding to an appliance maintaining a voltage-independent current i). One can imagine such an appliance as a strongly inductive load or as a small ideal motor with a huge flywheel, so that even large voltage changes do not produce any changes in i on the typical time scales of voltage fluctuations.

Let us briefly summarize the method used of Ref. [13]. Starting from van Kampen's approach [15], we consider first a discrete state space, where the states are numbered by the index n that corresponds, e.g., to the number of excess charge carriers (''electrons'' of charge ξ each) on the upper plate of the capacitor. This leads to the master equation for the charge of the capacitor, $q=n\xi$, which reads

$$\frac{dp_n}{dt}=-p_n(W_{n,n+1}+W_{n,n-1})+p_{n-1}W_{n-1,n}+p_{n+1}W_{n+1,n}, \tag{1}$$

with the transition rates $W_{n,n\pm1}=W^{(0)}_{n,n\pm1}+W^{(1)}_{n,n\pm1}+W^{(2)}_{n,n\pm1}$. We assume that electrons can pass from the lower to the upper plate of a capacitor or in the opposite direction through one of three independent channels, i.e., through the current generator (with the n-independent transition rates $W^{(0)}_{n,n\pm1}$, which are nonzero only between the states n and $n+1$ for $i>0$ and only between the states n and $n-1$ for $i<0$), or through one of the two diodes with the transition rates $W^{(1)}_{n,n\pm1}$ and $W^{(2)}_{n,n\pm1}$. Each of the two last channels satisfies the detailed-balance condition for its own temperature T, so that $W^{(1,2)}_{i,j}=W^{(1,2)}_{j,i}\exp[-(E_j-E_i)/kT_{1,2}]$. The energy of the system is determined by the capacitor's charge q and is equal to $E_n=(n\xi)^2/2C$. In the continuum limit one arrives at a Fokker-Planck equation for the probability distribution $p(u)$ of the voltage of the capacitor,

$$\frac{\partial p(u)}{\partial t}=\frac{\partial}{\partial u}\left\{\left[\left(\frac{1}{R_1(u)C}+\frac{1}{R_2(u)C}\right)u+i\right]p(u)+\left(\frac{kT_1}{R_1(u)C^2}+\frac{kT_2}{R_2(u)C^2}\right)\frac{\partial p(u)}{\partial u}\right\}. \tag{2}$$

Here the diodes are described as nonlinear resistors, whose volt-ampere characteristics are given by their differential resistance $R_i(u)$; see Ref. [13] for the details of the derivation. This differential resistance is connected with the transition rates through $R_i(q/C)=kT/W_i(q)\xi^2$, with $W_i(q)=W^{(i)}_{n,n+1}$. The meaning of Eq. (2) is quite transparent: The systematic part includes the overall current between the plates of the capacitor, consisting of the currents through the diodes and the nonfluctuating current through the outer device. The fluctuation term represents the contributions of the two diodes, each kept at its own temperature. Each contribution is connected to the corresponding systematic part via the corresponding fluctuation-dissipation relation. The stationary solution of Eq. (2) reads

$$p(u)=A\exp\left(-\int du\left\{\left[\left(\frac{1}{R_1(u)}+\frac{1}{R_2(u)}\right)u+i\right]\Big/\left(\frac{kT_1}{R_1(u)C}+\frac{kT_2}{R_2(u)C}\right)\right\}\right), \tag{3}$$

with A being the normalization constant. Note that under the no-current condition, $i=0$, and for equal temperatures $T_1=T_2=T$, the distribution $p(u)$ reduces to a Boltzmann distribution of the capacitor's energy, $p(u)=A\exp(-Cu^2/2kT)$, independent of the particular form of volt-ampere characteristics of nonlinear resistors.

The power of the engine is given by

$$P=iU, \tag{4}$$

with $U=\int_{-\infty}^{\infty}up(u)du$ being the mean voltage. The heat absorbed from the reservoir at temperature T_1 per unit time is given by

$$\dot{Q}=-\int u\left[\frac{kT_1}{R_1(u)C}\frac{\partial p(u)}{\partial u}+\frac{u}{R_1(u)}p(u)\right]du, \tag{5}$$

where the expression in brackets is exactly the (probability) current through the corresponding diode. We also note that, although Eq. (2) implies that $R(u)$ is sufficiently smooth, in final, integral expressions, Eqs. (4) and (5), we can change to piecewise-linear volt-ampere characteristics. Taking

$$R(u)=\begin{cases}R_+ & \text{for } u>0,\\ R_- & \text{for } u<0,\end{cases} \tag{6}$$

we arrive at

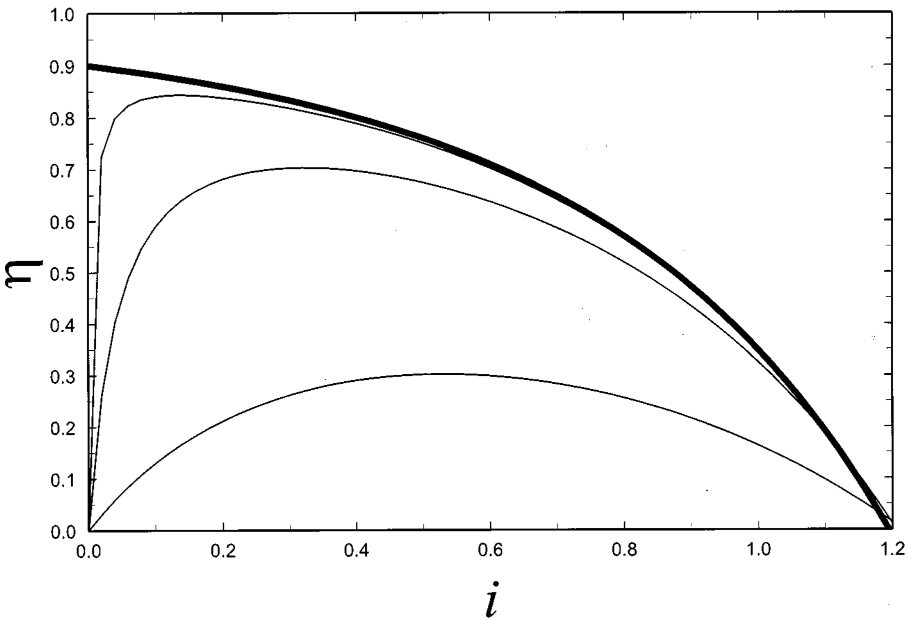

FIG. 2. Efficiency of the engine as a function of the (dimensionless) current i. The values of temperatures correspond to $T_1=10$ and $T_2=1$. Only the region where the positive work is produced is shown. The thick line corresponds to $R_-\to\infty$; the thin lines corresponds to $R_-=1000$, 100, and 10, from top to bottom. Note that the maximal efficiency for finite R_- is attained at a finite current, while for $R_-\to\infty$ it is attained under stalling condition $i=0$.

$$P(u)\propto\exp\left(-\frac{Cu}{2}\frac{2i+u/(R_++R_-)}{kT_1/[R_+\theta(u)+R_-\theta(-u)]+kT_2/[R_-\theta(u)+R_+\theta(-u)]}\right). \tag{7}$$

From Eq. (7) general expressions for P and Q and the efficiency $\eta=P/Q$ can be obtained in closed form. These general expressions are used for plotting Fig. 2, *vide infra*.

Let us first concentrate on the case of small currents. Let us consider a Taylor expansion of the voltage U in powers of i:

$$U(T_1,T_2,i)=U_0(T_1,T_2)+iU_1(T_1,T_2)+\cdots, \tag{8}$$

with $U_0(T_1,T_2)=U(T_1,T_2,0)$ and $U_1(T_1,T_2)=(d/di)U(T_1,T_2,i)|_{i=0}$. The engine's power is then given by $\dot{A}=iU_0(T_1,T_2)+O(i^2)$. The voltage for zero current reads

$$U_0(T_1,T_2)=\sqrt{\frac{2}{\pi}}(\tilde{T}_1-\tilde{T}_2)\frac{R_+^{-1}-R_-^{-1}}{\sqrt{R_+^{-1}+R_-^{-1}}(\sqrt{R_+^{-1}\tilde{T}_1+R_-^{-1}\tilde{T}_2}+\sqrt{R_-^{-1}\tilde{T}_1+R_+^{-1}\tilde{T}_2})}, \tag{9}$$

where we have introduced $\tilde{T}_i=kT_i/C$. On the other hand, for the absorbed heat per unit time $\dot{Q}$, one has

$$\dot{Q}(T_1,T_2,i)=Q_0(T_1,T_2)+iQ_1(T_1,T_2)+O(i^2), \tag{10}$$

with $Q_0(T_1,T_2)=Q(T_1,T_2,0)$ and $Q_1(T_1,T_2)=(d/di)Q(T_1,T_2,i)|_{i=0}$. The values of these coefficients are

$$Q_0(T_1,T_2)=\frac{\tilde{T}_1-\tilde{T}_2}{R_++R_-} \tag{11}$$

(which, remarkably, is linear in the temperature difference) and

$$Q_1(T_1,T_2)=\sqrt{\frac{2}{\pi}}\tilde{T}_1\frac{R_+^{-1}-R_-^{-1}}{\sqrt{R_+^{-1}+R_-^{-1}}(\sqrt{R_+^{-1}\tilde{T}_1+R_-^{-1}\tilde{T}_2}+\sqrt{R_-^{-1}\tilde{T}_1+R_+^{-1}\tilde{T}_2})}. \tag{12}$$

Note that $Q_1(T_1,T_2)$ is proportional to $U_0(T_1,T_2)$, and that the proportionality constant is exactly $\eta_{\rm Carnot}=(T_1-T_2)/T_1$. Let us now consider the efficiency of the engine at small currents. It is equal

$$\eta\approx\frac{iU(T_1,T_2)}{Q_0(T_1,T_2)+iQ_1(T_1,T_2)}=\eta_{\rm Carnot}(1+Q_0/iQ_1)^{-1} \tag{13}$$

and tends to zero for small currents unless $Q_0(T_1,T_2)$ vanishes. This fact shows that the irreversible mode of the operation of an engine is really due to the fact that it transports heat under a no-current condition. On the other hand, from Eq. (11) one readily infers that Q_0 tends to zero for $R_-\to\infty$. In this last case $\eta=\eta_{\rm Carnot}=(T_1-T_2)/T_1$ for $i\to 0$. Thus, the appliance using ideal diodes with infinite backward resistance does not transfer heat under the no-work condition and achieves the Carnot efficiency.

One can also put this finding into a broader context of nonequilibrium thermodynamics. The fluctuation rectification in our system is a kind of thermoelectric effect. For moderate temperature differences ΔT, one can describe it within a linear approximation, leading to $U=a_{11}i+a_{12}\Delta T$, where the coefficients a_{ij} depend on the mean temperature T and on the parameters of the diodes and of the capacitor. On the other hand, the heat absorbed from the hot reservoir is given by $\dot{Q}=a_{21}i+a_{22}\Delta T$. The Onsager principle suggests that nondiagonal coefficients a_{12} and a_{21} are coupled. From Eqs. (9) and (12) it follows indeed that $a_{21}=a_{12}\tilde{T}$. On the other hand, the no-current heat transport leading to irreversibility is determined by a diagonal term a_{22}. This coefficient can be arbitrarily reduced by tuning the parameters of the system while keeping the nondiagonal coefficients finite.

The overall behavior of the efficiency as a function of current i is shown in Fig. 2. Here the efficiency for $\tilde{T}_1=10$, $\tilde{T}_2=1$, and $R_+=1$ is evaluated numerically using Eqs. (3), (4), and (5). The thick curve corresponds to the case $R_-\to\infty$ and reaches for $i\to 0$ the Carnot value of 0.9. Three other curves correspond to the values of $R_-=1000$, 100, and 10, respectively. Note that for $R=10$ the overall efficiency curve differs considerably from the efficiency for the ideal rectifier; for larger R the large-current efficiencies of an ideal and of a nonideal system start behaving in a similar way; at lower currents the efficiency of a nonideal appliance departs from the ideal curve and rapidly tends to zero. For large R_- the maximal efficiency is achieved at rather small currents and can be very close to the limiting, Carnot value. On the other hand, to achieve the efficiencies close to Carnot's ones for fixed temperatures one really needs a very large reverse resistivity (say, of the order of 10^3 of the direct one).

We have considered the efficiency of a heat engine (using a fluctuation rectification principle), employing two diodes switched in the opposite directions and kept at different temperatures. We show that this engine (contrary to a Feynman ratchet and to a similar electric appliance using a linear resistor instead of a second diode) can achieve the Carnot efficiency when ideal diodes with infinite resistances in the backward direction are applied. This shows that there exist no principal thermodynamic bounds on the efficiency of a heat engine that is in contact with two heat baths simultaneously.

The hospitality of LMHD at the University Paris VI and the financial support by the CNRS are gratefully acknowledged. The author is indebted to Professor J. Prost for helpful discussions.

[1] F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997).
[2] R. D. Astumian, Science **276**, 917 (1997).
[3] M. Bier and R. D. Astumian, Bioelectrochem. Bioenerg. **39**, 67 (1996).
[4] P. Hänggi and R. Bartussek, *Nonlinear Physics of Complex Systems-Current Status and Future Trends*, edited by J. Parisi, S. C. Müller, and W. Zimmermann, Lecture Notes in Physics Vol. 476 (Springer, Berlin, 1996), pp. 294–308.
[5] C. R. Doering, Nuovo Cimento D **17**, 685 (1995).
[6] K. Sekimoto, J. Phys. Soc. Jpn. **66**, 1234 (1997).
[7] J. M. R. Parrondo, Phys. Rev. E **57**, 7297 (1998).
[8] J. M. R. Parrondo, J. M. Blanco, F. J. Cao, and R. Brito, Europhys. Lett. **43**, 248 (1998).
[9] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1963), Vol. 1.
[10] I. M. Sokolov and A. Blumen, J. Phys. A **30**, 3021 (1997); Chem. Phys. **235**, 39 (1998).
[11] P. Reimann, R. Bartussek, R. Häussler, and P. Hänggi, Phys. Lett. A **215**, 26 (1996).
[12] J. M. R. Parrondo and P. Español, Am. J. Phys. **64**, 1125 (1996).
[13] I. M. Sokolov, Europhys. Lett. **44**, 278 (1998).
[14] L. Brillouin, Phys. Rev. **78**, 627 (1950).
[15] N. G. van Kampen, J. Math. Phys. **2**, 592 (1961).